Driven Boson Sampling

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Sampling the distribution of bosons that have undergone a random unitary evolution is strongly believed to be a computationally hard problem. Key to outperforming classical simulations of this task is to increase both the number of input photons and the size of the network. We propose driven boson sampling, in which photons are input within the network itself, as a means to approach this goal. We show that the mean number of photons entering a boson sampling experiment can exceed one photon per input mode, while maintaining the required complexity, potentially leading to less stringent requirements on the input states for such experiments. When using heralded single-photon sources based on parametric down-conversion, this approach offers an $\sim e$-fold enhancement in the input state generation rate over scattershot boson sampling, reaching the scaling limit for such sources. This approach also offers a dramatic increase in the signal-to-noise ratio with respect to higher-order photon generation from such probabilistic sources, which removes the need for photon number resolution during the heralding process as the size of the system increases.

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Boson sampling [1] is strongly conjectured to be a computationally hard problem. It describes the sampling from the output distribution of indistinguishable bosons evolving through a sufficiently large random unitary, as depicted in Fig. 1(a). While not a universal quantum computation problem, such as linear optics quantum computing [2], boson sampling has attracted considerable attention due to its experimental feasibility with quantum optics. Different photonic platforms have demonstrated inputting up to 4 single photons in networks of up to 13 input modes [3–11]. However, it remains a challenge to scale up the devices to 20–30 photons [1] traversing a correspondingly large network, a regime in which a quantum boson sampling machine is expected to outperform classical computers.

In the first boson sampling experiments [3–6], parametric down-conversion (PDC) sources were employed and thus the photons were generated in a probabilistic fashion. With this scheme, measurement time scales exponentially with photon number. To improve this performance, two complementary approaches have been developed. Recently, source hardware has been improved by implementing quasi-on-demand single-photon sources as inputs to a boson sampling circuit [10,11], resulting in a significant reduction in measurement time. In parallel, algorithmic ("software") developments have also improved the scaling when using probabilistic sources. Scattershot boson sampling (SBS) [8,12,13] increases the number of possible inputs to the linear network, as shown in Fig. 1(b), by a binomial factor, which reduces the measurement time by a corresponding amount. However, probabilistic PDC sources typically suffer from the additional limitation of high-order photon contributions. In general, as the required number of photons increases, the chance of higher-order terms also increases.

If the number of *possible* inputs can be increased compared to the number of photons, the rate is increased while the effect of higher-order terms arising from PDC can be reduced. This is because the pump power of each source is reduced without lowering the overall generation probability. In SBS, the number of input modes defines one dimension of the network, which in turn determines the required depth of the network. Therefore, arbitrarily increasing the number of possible inputs necessarily increases the network size in both width and depth, which squares the number of required components. Thus the question arises: can the number of possible inputs be decoupled from the width, such that sufficiently many possible inputs can be constructed without blowing up the network size?

To answer this question, we propose driven boson sampling (DBS) as a means to increase the number of possible inputs when using heralded PDC photon sources, independent of the size of the network. This approach increases the input state generation rate and significantly reduces the effect of higher-order photon contributions, overcoming the need for high-efficiency photon-number resolving herald detectors [14]. In our scheme, we consider a stack of SBS-type experiments as shown in Fig. 1(c), in which the output of one SBS experiment becomes the input to the next. Additional photons can be injected in any of the k input layers, and undergo evolution through a series of independent $m \times m$ Haar-random unitaries $\mathbf{U}_H^{(k)}$. In general, the requirements on size imply $m \gg n^2$ in order to reduce the chance of multiple photons in the output modes

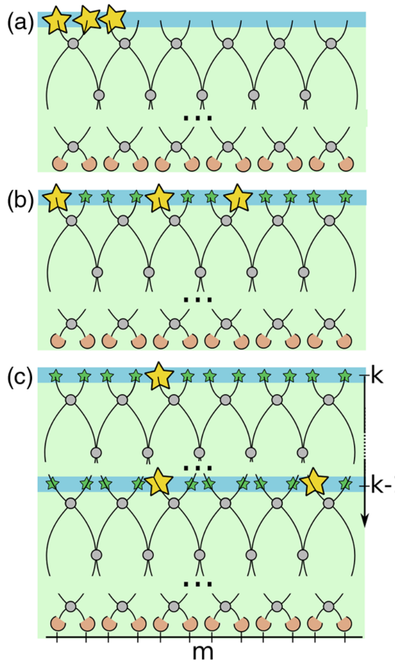

FIG. 1. Boson sampling networks for (a) boson sampling, (b) scattershot boson sampling, and (c) driven boson sampling with the state generation governed by a stack of k input layers each followed by an $m \times m$ Haar-random unitary matrix $\mathbf{U}_H^{(k)}$. Green stars indicate all possible modes for injecting photons, yellow stars mark one possible ($n = 3$) input state.

(overcoming the so-called birthday problem [1]). It has been shown that, in the case of exact boson sampling, the depth of each unitary network $\mathbf{U}_H^{(k)}$ need not exceed 4 layers of beam splitters to be classically hard [15]. However, for the case of approximate boson sampling, the minimum depth bounds are $\mathcal{O}(n \log m)$ [1] and $\mathcal{O}(m \log m)$ [13] for standard and scattershot boson sampling, respectively. This depth requirement also arises from the number of beam splitters required to implement an arbitrary unitary [16,17], of which a Haar-random unitary is just one example.

It may appear that by injecting bosons within a network we move away from the fundamental constraint of unitary dynamics to the nonlinear regime, which is not covered under existing hardness conjectures of boson sampling. However, this scheme can in fact be mapped to a valid boson sampling problem. To illustrate, we begin with the abstraction shown in Fig. 2. The input modes of each unit can be extended to the top of the network [as shown in Fig. 2(b)], creating an input state vector of length $k \cdot m$ at the top of the network, similar to the SBS case. We then write the evolution of the whole system as a transformation of an input state of length $k \cdot m$ to an output state of length m, via the $k \cdot m \times m$ scattering matrix $\mathbf{G}$.

The input state is

$$|S_{\text{in}}\rangle = \bigotimes_{i=1}^{k \cdot m} (a_i^\dagger)^{s_i} |0_i\rangle = |s_1, \ldots, s_{k \cdot m}\rangle, \qquad (1)$$

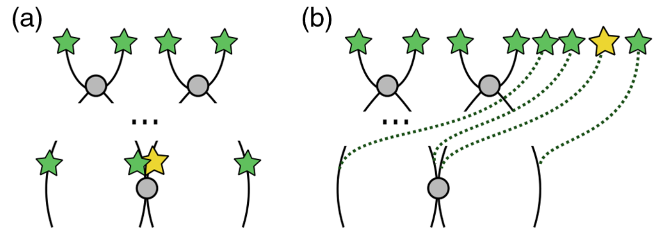

FIG. 2. (a) State generation in driven boson sampling. (b) Equivalent system with an adapted graph (green dashed lines) and single-photon input state. Green stars indicate all possible modes for injecting photons, the yellow star marks an equivalent input position.

where $a_i^\dagger$ is a bosonic creation operator in mode i and $s_i \in \{0, 1\}$ describes single photons in n of the $k \cdot m$ modes and vacuum otherwise. After the evolution governed by $\mathbf{G}$ and projective measurement, the measurement outcome $|S_{\text{out}}\rangle = |t_1, \ldots, t_m\rangle$ with $t_i \in \{0, 1\}$ is related to the permanent of an $n \times n$ submatrix $[\mathbf{G}]^{(S_{\text{out}}|S_{\text{in}})}$ (the elements of which are visualized by the intersection of the orange rows and the blue columns in Fig 3), following the procedure in, e.g., Ref. [18], such that the probability of a particular outcome $|S_{\text{out}}\rangle$ given an input state $|S_{\text{in}}\rangle$ is related to the permanent by

$$P(S_{\text{out}}|S_{\text{in}}) \propto |\text{Per}([\mathbf{G}]^{(S_{\text{out}}|S_{\text{in}})})|^2. \qquad (2)$$

While it is long understood that calculating permanents of matrices is hard [19], the insight from Aaronson and Arkhipov was to show that efficient *sampling* from distributions governed by the permanents of $n \times n$ Gaussian matrices contained within an $n \times m$ scattering matrix would have profound implications for the hierarchy of computational complexity. It is therefore strongly conjectured to be a #P-hard problem (the permanent-of-Gaussians conjecture [1]), even in the approximate case where we allow for errors. SBS [12] extends the size of the scattering matrix to $m \times m$, and samples an ensemble average of n photons in all possible m inputs, which yields an $\binom{m}{n}$ increase in the input state generation rate. In DBS, the scattering matrix $\mathbf{G}$ is now of size $k \cdot m \times m$, yielding an enhancement input state generation proportional to $\binom{k \cdot m}{n}$.

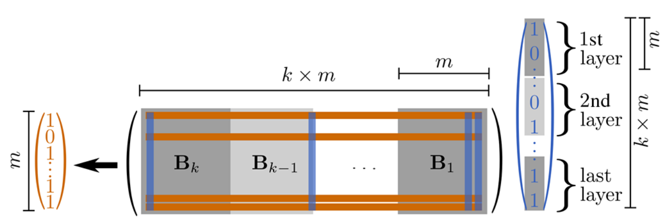

FIG. 3. Pictorial relationship of the input state to a particular measurement outcome $|S_{\text{out}}\rangle$ due to matrix $\mathbf{G}$ built from k blocks B_i of size $m \times m$ [18].

To provide strong evidence for the complexity of this problem, we show that the $n \times n$ submatrices which govern the evolution of a single instance of this DBS machine remain close in variation distance to a matrix of independent and identically distributed (i.i.d.) Gaussians, in line with Theorem 3 of the original hardness conjecture [1]. This states that a sufficiently small sample of elements from a Haar-random matrix contains insufficient structure to efficiently compute the permanent, i.e., that those elements are close in variation distance to a matrix of i.i.d. Gaussian elements [1]. The submatrix $[\mathbf{G}]^{(S_{\mathrm{out}}|S_{\mathrm{in}})}$ sampled by our photons comprises elements from different, independent Haar-random blocks (see gray blocks in Fig. 3); therefore, the elements of this matrix are at least as independent as elements sampled from a single $m \times m$ Haar-random unitary.

To illustrate, we consider the block $\mathbf{B}_1$, which describes the evolution after the final photon generation layer. This is built from products of $m \times m$ unitary coupling matrices $\mathbf{C}_i^{(1)}$ (see Supplemental Material for the exact form of the $\mathbf{C}_i$ matrices [20]), the elements of which are chosen such that the block $\mathbf{B}_1$ is Haar random, i.e., $\mathbf{B}_1 = \prod_{i=1}^m \mathbf{C}_i^{(1)} = \mathbf{U}_H^{(1)}$. The preceding block $\mathbf{B}_2$ is constructed from coupling matrices $\mathbf{C}_i^{(2)}$ in a similar manner, but it is also multiplied by the first block, i.e., $\mathbf{B}_2 = (\prod_{i=1}^m \mathbf{C}_i^{(2)})\mathbf{B}_1 = \mathbf{U}_H^{(2)}\mathbf{U}_H^{(1)} = \mathbf{U}_H^{(2)\prime}$. Thus, the qth block is $\mathbf{B}_q = (\prod_{i=1}^q \mathbf{U}_H^{(i)}) \cdot (\prod_{j=1}^m \mathbf{C}_j^{(q)}) = \mathbf{U}_H^{(q)\prime}$.

This continues up to k input layers, such that the evolution of photons generated in each layer is governed by an independent random matrix. Thus, sampling the probability distribution arising from the submatrices governed by elements from these independent random blocks retains at least the level of complexity as the original boson sampling problem. Moreover, the complexity proofs for sampling from an ensemble of these matrices (i.e., SBS) must also apply in this case.

One important consequence of our result is that it allows input states with more than one photon on average. If one considers the case where $n-1$ photons are generated in the penultimate layer, there are $(n-1)/m$ photons, on average, at each input mode in the final layer; the photons that have been generated are distributed across all the modes. In the final generation layer, one of these modes picks up the final photon, such that the final unitary has as an input state $1+(n-1)/m$ photons in one mode [and $(n-1)/m$ in the others]. This is still a valid input state, despite being a state of noninteger photon number. An intuitive explanation for this surprising result is that the first $n-1$ photons have lost any potential information following propagation through the first random unitary; therefore, they can yield no extra information about evolution through the next unitary.

Furthermore, while we have demonstrated the complexity of this scheme for m layers of beam splitters between input layers (required to implement the Haar-random unitary), it remains an open question whether this depth requirement can

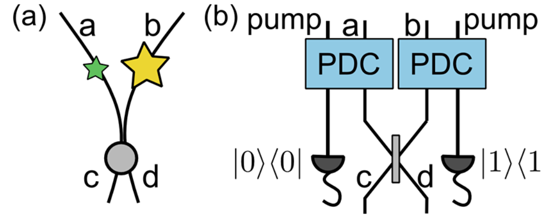

FIG. 4. Fundamental unit of the network (a) photon creation at a link b and no photon created at link a. (b) Experimental implementation using heralded parametric down-conversion.

be reduced. We note further that each instance of a DBS experiment in which photons are generated in the same layer of the generation network corresponds to a SBS problem. However, these SBS instances are an exponentially small subset of the DBS problem.

To demonstrate the benefits of our scheme, we consider an experimental approach which is readily implemented using heralded parametric down-conversion, as shown in Fig. 4(b). Measuring a single photon heralds the presence of a new photon within the network [20]. Adding photons in this manner increases the total number of input modes in $\mathbf{G}$ to $k \cdot m$. It is necessary that each source can be heralded, such that it is known within each trial how many sources fire.

In the original boson sampling scheme, single photons are input in predetermined positions, specifying a single configuration of modes with and without photons. If one uses n heralded single-photon sources at the input, for example, arising from PDC, one must wait for all n heralds before a boson sampling experiment can commence. This occurs with probability $P_s^{\mathrm{BS}}(n) = P_1^n$, where P_1 is the single-photon generation probability for each source. In SBS, all m input modes are coupled to heralded single-photon sources. However, all possible configurations of exactly n of the m sources firing is a valid input state; therefore, one gains an m choose n speed-up in the number of valid trials, whereby $P_s^{\mathrm{SBS}}(n) = \binom{m}{n} P_1^n P_0^{m-n}$. Here, P_0 is the probability of no photons (vacuum) being generated. In DBS, a valid generation event of n single photons occurs with success probability $P_s^{\mathrm{DBS}}(n) = \binom{k \cdot m}{n} P_1^n P_0^{k \cdot m - n}$, where $k \cdot m$ is the number of possible input positions.

The advantage offered by DBS is demonstrated by optimizing the single-photon generation probability for a desired photon number n. For PDC states of the form $|\psi_{\mathrm{PDC}}\rangle = \sqrt{1-\lambda^2} \sum_{i=0}^{\infty} \lambda^i |i,i\rangle$, with i the photon number, the probability of generating a photon is $P_1 = (1-\lambda^2)\lambda^2$, and vacuum $P_0 = (1-\lambda^2)$, where λ is the squeezing parameter. For fixed photon number n, and number of possible inputs $k \cdot m$, we can find the optimal λ to maximize success probability $P_s(n)$:

$$\lambda_{\mathrm{opt}} = \sqrt{\frac{n}{k \cdot m + n}}. \quad (3)$$

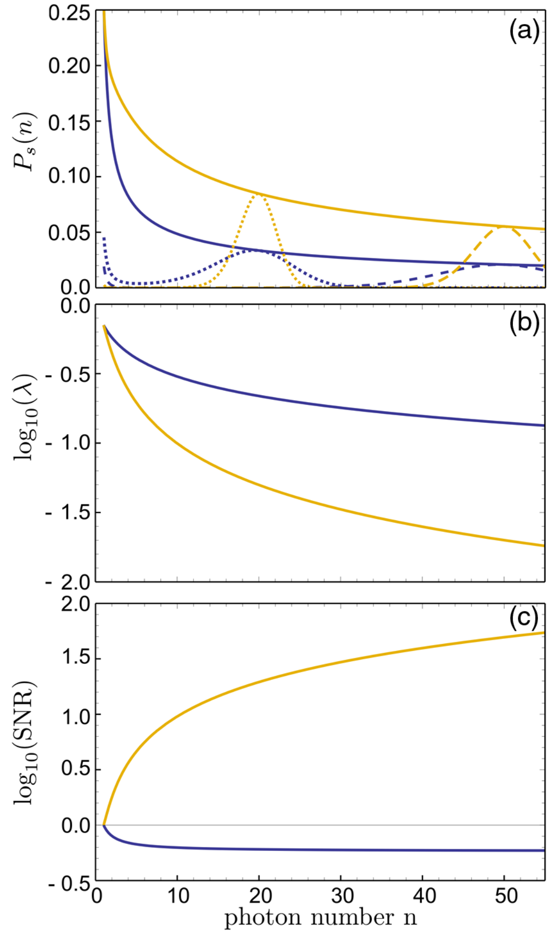

FIG. 5. Comparison of DBS (orange) and SBS (blue) as a function of photon number. (a) Success probabilities $P_s(n)$ for optimal generation efficiencies λ (solid lines) as well as two example distributions of $P_s(n)$ for optimized for $n = 20$ (dotted line) and $n = 50$ (dashed line). (b) Optimal generation efficiency λ for an n photon event. (c) Signal-to-noise ratio (SNR) of heralding and generating single-photon events divided by the probability of higher-order contributions (see Supplemental Material [20]). In this particular example, the DBS case corresponds to $k = n$, and the SBS corresponds to $k = 1$. In both cases we assume $m = n^2$.

In DBS we are free to choose the number of layers k under the constraint $k \cdot m \geq n^2$. Note that $k = 1$ is the case for SBS. Following the Supplemental Material of Ref. [12], in the asymptotic limit for the number of possible sources $k \cdot m \geq n^2$, the scaling of optimal generation probability is $P_{\max} \sim (1/b\sqrt{2\pi})(1/\sqrt{n})$, where, for $m = n^2$, the factor $b \to e^{1/k}$ in the limit of $n \to \infty$. Thus, for large k, the geometry of DBS allows the success probability $P_s(n)$ to approach a factor of e higher compared to SBS [Fig. 5(a)], enabling more than twice the data rate of scattershot boson sampling for fixed laser repetition rate (or, equivalently, more than twice the acquired data for a fixed experiment time).

Perhaps more significantly than a constant factor speed-up is the dramatic reduction in the optimal squeezing parameter λ to achieve this improvement [Fig. 5(b)]. By choosing, for example, $k = \sqrt{m} = n$, the optimal λ reduces by $\lesssim 2$ orders of magnitude. Not only does this reduce pump power requirements for the $k \cdot m$ sources, but also the probability of generating higher-order terms which act as noise sources on the signal, from which we calculate the signal-to-noise ratio (SNR). In order to achieve a SNR > 1, we find that the minimum number of layers when $m = n^2$ is $k \geq \lceil 1/n(\sqrt{n}2 - 1) \rceil$ (see Supplemental Material [20]). In fact, the SNR exceeds unity for all $k \geq 2$, independent of n. This means that large numbers of heralded single photons can be generated while higher-order contributions are almost completely suppressed, thus overcoming the need for photon-number-resolving detectors. Indeed, the SNR for DBS actually increases with photon number [Fig. 5(c)], which makes PDC sources used in this manner a promising candidate for scaling up boson sampling experiments.

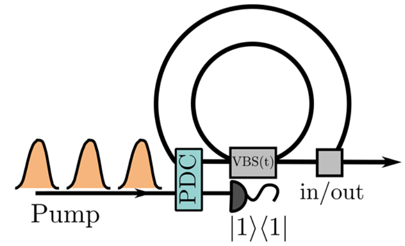

FIG. 6. Scheme of the time-multiplexing setup with a PDC source within the loop, adapting the scheme by Ref. [22]. The variable beam splitter VBS(t) implements all nodes in the network.

Although DBS significantly improves the measurement rate of a boson sampling experiment, this is at a cost of increased input sources s (from originally $s = n$ sources to $s = m$ sources in SBS, to $s = k \cdot m$ sources in DBS). The operation of many sources of indistinguishable photons is a challenging task and the timing of the additional inputs is crucial such that all photons may interact, independent of the position where they are generated. Furthermore, the additional depth of the network increases from m to $k \cdot m$ layers of beam splitters. However, employing techniques from time-multiplexed quantum networks [21] inherits all the benefits of photon indistinguishability and homogeneity while simultaneously reducing the physical overhead to a single set of components. Indeed, within the context of boson sampling, such a loop architecture has been proposed [22] and experimentally demonstrated [11].

DBS is easily adapted to this approach by placing a down-conversion source within the loop structure, as shown in Fig. 6. The timings of the sources can easily be determined by lengths of the fiber loop which correspond to the repetition rate of a mode-locked pump laser. The optimal number of source layers k will depend on the overall transmission through the network, although $k > 2$ is sufficient to improve upon SBS.

In conclusion, we propose driven boson sampling to improve the generation rate of valid input states while reducing the necessary pump powers per source significantly. The reduction of pump power drastically decreases the impact of higher-order photon contributions and improves the SNR, demonstrating our approach as a promising candidate to scale up boson sampling machines. Furthermore, the concept of placing sources of quantum light within a quantum network remains a largely unexplored area. We have demonstrated the benefits of this technique to the specific example of boson sampling, but elements of this approach may well find applications in a range of quantum optics protocols.

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